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Invisibility Cloaking

via

Coordinate Transform Optics

(A review of the literature covering Invisibility Cloaking with specific reference to Triangular Craft)

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Coordinate Transform Optics (CTO) is a methodology for tailoring waves to achieve certain desirable characteristics. It is achieved via the use of smooth differentiable coordinate transforms. The mathematically defined transformations are then cast into corresponding changes of constituent material parameters to achieve the desired functionality. Although this paper will be focused on the visible Electro-Magnetic (EM) spectrum (invisibility), the procedure works for any set of linear waves whose equations are invariant under coordinate transformations (Sonar, Heat, Radio etc.). To the author's knowledge, the first demonstration of this was by Pendry et. al. in 2006[1].

There are actually two closely related methods of CTO in use today: Transformational Optics (TO), and Conformal[2] Mapping (CM). Both methods rely on the form invariance of Maxwell's Equation under coordinate transformations. TO utilizes a general metamaterial[3] anisotropic[4] layer on which it develops a spatial map of the electric permittivity[5] (ε) and magnetic permeability[6] (μ) to achieve its result. CO utilizes an isotropic[7] dielectric[8] medium on which it develops a spatial map of the Refractive Index[9] distribution to achieve its result.

The use of general coordinate transformations and the more general meta-material media in the TO methodology provides it with a much wider utility than is found with the CM methodology. Therefore most invisibility results found in the literature today are those using TO generated cloaks. This method starts with a defined functionality and shape. The designer then uses those inputs to define a transformation that will produce them. The transformation is applied to Maxwell's equations and its constituent equations and a spatial map of the electric permittivity (ε) and the magnetic permeability (μ) required to cause the desired warped trajectories is determined. The difficult part of building the material is then attempted. It should be noted that presently TO has only been proven for single frequencies in the microwave and IR ranges. Broadband results for the visible have not yet been accomplished.

Although not at all obvious, it may be noted that TO designs are not unique. Using the above methodology, TO designs can always be found for any desired functionality. In fact, in most cases there will be an infinite set of designs that would accomplish the specific function with each design only differing in the material used to accomplish it. The problem is thus not determining the spatial distributions but an ability to feasibly construct the material needed. We, however, are in a position similar to Physicists when they have to solve a particularly hard mathematical problem. Unlike Mathematicians, Physicists never attempt to prove a solution to the problem exists. They argue that if the equation describes nature correctly, the solution must exist. In our case, we need not worry about the complexity of any design. We should argue that if we can describe what we see accurately enough, it must exist.

From the above, it is obvious that the CM methodology is much more restrictive than the TO methodology. In the CM methodology, the transformation must be Conformal; and it uses the simpler isotropic dielectric materials. Therefore, if a CM solution is found, it will scale much more easily with size and wavelength. It would also have more of a broadband character than a TO solution. Like the TO methodology, CM also starts with a defined shape and functionality.. CM then uses the fact that like Maxwell's Equations, the 2-dimensional Helmholtz Equation for plane waves is invariant under Conformal coordinate transformations. Thereafter in the CM methodology, one then attempts to find a conformal transformation that will satisfy the functionality requirements desired with the defined shape.

For reasons only of interest to Mathamaticians[10,11], finding a transformation that establishes a specific CM mapping of domains (shapes) is often impossible. The method is therefore restricted in both material and shape. Additionally, since it is much easier to arrange for a surface to absorb a wave rather than warp it around an object, the CM method has mostly been restricted to stealth technology. Recently[12,13] (2010-2011) however, it has been shown possible (via a combination of TO and quasi-CM) to construct an "invisibility carpet[14] operating over the entire visible domain from naturally occurring bi-refringent calcite crystals. The crystals have overcome:

- the energy absorbing tendency of meta-material metal cloaks in the visible light range;
- the small size limitation of meta-material cloaks (the calcite cloaks hide areas of cm size);
- the costliness of meta-material fabrication; and
- the limitation of meta-material cloaks to only cloak a single frequency.

The major limitation found in theses cloaks is their ability to only operate in a single wave mode[15] (TE or TM). This, however, may be an easier problem to overcome then the problems associated single wavelengths (frequencies).

2. Transformational Optics

As stated above, Transformational Optics is based on the mapping of Maxwell's Equations from one coordinate set to another. Since the EM spectrum (light etc.) does not care how we look at it, these equations must be invariant under any acceptable spatial coordinate transformation. Any transformation between sets of 3-d coordinates can be written as the new set of variables as a function of the old set. Specifically transforming from the standard Orthogonal Cartesian (x, y, z) system to a new one (x'_I) can be written as the following 3 equations:

$$x'_i = f_i(x, y, z). \qquad \{ i = 1,2,3 \}$$

It may be noted that, use of the Cartesian system was only used as an example.

In a medium with no sources or currents, Maxwell's Equations are given by:

(Curl Equations) $\nabla \times \mathbf{E} = -(\partial \mathbf{B} / \partial t) / c$, $\nabla \times H = -(\partial \mathbf{D} / \partial t) / c$,
(Divergence Eqs.) $\nabla \cdot \mathbf{D} = 0$, $\nabla \cdot \mathbf{B} = 0$

In a linear medium the constituent relations are:

$$D = \varepsilon E, \qquad B = \mu H;$$

where ε and μ, the dielectric permittivity and magnetic permeability, are normally tensors describing the transmitting material (usually air). If we transform from coordinates $\{ x_i \}$ (i = 1,2,3) to coordinates $\{ x'_k \}$ (k = 1,2,3) neither the form of Maxwell's Equations nor that of the constituent relations will change. The objects that will change are the elements of the dielectric permittivity and magnetic permeability tensors.

The above transformation can be written in terms of a matrix of first order partial derivatives $\{J^k_i = \partial x^k / \partial x_i \}$ called the Jacobian. Once one has calculated the nine element of this matrix { **J** } the transformation of vectors E and **H** can be written.

$$\mathbf{E'}_{new} = \mathbf{J} \cdot \mathbf{E}_{old} \qquad \mathbf{H'}_{new} = \mathbf{J} \cdot \mathbf{H}_{old}$$

$$E'^{k}_{new} = J^k_1 E_1 + J^k_2 E_2 + J^k_3 E_3 \quad \text{and} \quad H'^{k}_{new} = J^k_1 H_1 + J^k_2 H_2 + J^k_3 H_3$$

Similarly, the Permittivity and Permeability tensors transform as:

$$\boldsymbol{\varepsilon'} = \mathbf{J}\, \boldsymbol{\varepsilon}\, \mathbf{J}^{T'} / \det(J) \quad \text{and} \quad \boldsymbol{\mu'} = \mathbf{J}\, \boldsymbol{\mu}\, \mathbf{J}^{T'} / \det(J).$$

(In these equations the superscript T indicates the transpose of matrix J and det(J) indicates the determinant.) It is emphasized that the set { **E, H, ε, & μ** } represent the wave external to the cloaking medium and the set { **E', H', ε', & μ'** } represent the wave in the cloaking medium.

To cloak arbitrary shaped machines it would be nice to have a cloak that would conform to the shape. That, however, is still in the future. At present, there are only 3 families of "perfect" (actually none of them are perfect) TO cloaks being designed. These are:

- 3-d point cloaks
- 2^+-d line cloaks.
- 3-d point and line cloak combination, and
- 2-d carpet cloaks

A point cloak is any cloak through which all rays, from any direction go though a single point. Among others, this would include spherical and ellipsoidal cloaks. It may be noted that if one look at an object hidden by a broadband visible point cloak, one would see exactly what is behind the object in the exact place they would be if the cloaked object were not there.

A line cloak is any cloak through which all EM rays pass through a line and there must be at least 1 point in the interior from which all points in the shell can be reached by a straight line. Cylindrical cloaks are among this type of cloak. These cloaks are not considered 3-d since the ends of the cylinder do not fit the definition and are not included. Assuming one could look at an infinitely long cylindrically broadband visibly cloaked object, one would also see what is behind the cloaked object and, as with the point cloak, they would still be in the exact place they would be if the cloaked object were not there. The viewing problem would occur at the ends. Due to the derivative discontinuity the rays which pass through the discontinuity an outline of it will be seen but would probably be very faint. (This will be investigated further later in this discourse.) Additionally dependent on how the cloaking of the ends was made the object behind the ends may change location. Some change in location would also occur if the ends were the half hemispheres discussed above.

3. TO Example (Spherical Invisibility Cloak[25])

As stated above, a spherical cloak is a version of the point cloak. Due to the relative simplicity of spherical coordinates, this is one of the easier volumes to cloak via TO. A three-dimensional spherical cloak is constructed by compressing the EM fields of a spherical volume into a spherical shell defined by an inner radius { R_{in} } and an outer radius { R_{out} }. Let r' = f(r) represent the general transformation desired. Since we wish the new coordinates to follow the shell we now have the boundary conditions:

$$f (R_{out}) = R_{out} \qquad \text{and} \qquad f (R_{in}) = 0.$$

Letting { $\mathbf{R} = (R_{out} - R_{in}) / R_{out}$ } this transformation in terms of spherical components is:

$$\mathbf{r'} = \mathbf{R}\, r + R_{in} \qquad \& \qquad \varphi' = \varphi, \quad \theta' = \theta \qquad (R_{out} \geq r \geq 0)$$

and $\quad r = \{ r' - R_{in} \} / \mathbf{R} \qquad \& \qquad \varphi = \varphi', \quad \theta = \theta\ni.$ {the inverse transform}

If we could simply insert derivatives of these components to create the Jacobian, the answer would be a relatively simple.[16] The interesting result is although one must account for the vector nature of the transform ignoring it in this instance actually provides the correct answer. The reason for that is the entire problem is spherically symmetric. Therefore one can evaluate terms at any point and not lose generality. Specifically we will be evaluating at: $\mathbf{R'}^T = \{ r', 0, 0 \}$. Since at that point, the problem is actually a scalar one rather than a vector one it provides the same result.

As stated above the vector equations can be skipped for this problem. However to be complete, they have been included. Allowing **R** to represent vector **r,** (and since $|\mathbf{R}| = r$) the vector form of the transform is:

$$\mathbf{R'} = \{ \mathbf{R} + R_{in} / r \} \mathbf{R}.$$

The Jacobian matrix in terms of the original coordinates is then:

$$\mathbf{J} (\mathbf{r}) = \{ \mathbf{R} + R_{in} / r \} \boldsymbol{1} - (R_{in} / r^3) \mathbf{XX^T}.$$

(In this equation, ***1*** represents the identity tensor[17] and ($\mathbf{XX^T}$ }[18] represents the tensor product of vectors **X** and **X**.) As is to be expected in a spherically symmetric problem, the above transformation leaves radial unit vectors relationships unchanged. Therefore: we can write the relationship:

$$\{\text{radial unit vector}\} = \mathbf{R} / r = \mathbf{R'} / r',$$

The Jacobian in the new coordinates is then:

$$J (\mathbf{r'}) = \{ r' \boldsymbol{1} - [R_{in} / r'^3] \mathbf{R'R'^T} \} / r .$$

(It is known that is not absolutely a function of only r'. One could multiply through by r to make it absolutely a function of r' but it will ignore that for now.)

As stated above, it is known the problem is spherically symmetric and one can therefore evaluate **J** at any point on the sphere without loss of generality. Therefore we will concentrate on the point where both angles are 0 { $\mathbf{R'}^T = [r', 0, 0]$ }. At this point both J(r) and **J** (r') are diagonal[19]; $\mathbf{J} (\mathbf{r'}) = \mathbf{diag} \{ (r'_t - R_{in}), r', r' \} / r$ **.**

Without proof, I will state the determinant of the above Jacobian is the product of the 3 elements.

$$\det (\mathbf{J}) = (r'_t - R_{in}) r'^2 / r^3$$

After a little algebra this can be reduced to:

$$\det(\mathbf{J}) = \{ r'^2 (R_{out} - R_{in})^3 \} / \{ (r' - R_{in})^2 R_{out}^3 \} .$$

The values for the permittivity and permeability can now be determined from

$$\boldsymbol{\alpha}' = \mathbf{J}\, \boldsymbol{\alpha}\, \mathbf{J}^{T'} / \det(\mathbf{J}) = \boldsymbol{\alpha}\, \mathbf{J}^2 / \det(J),$$

where **α** stands for either of the tensors. Since all of these tensors are diagonal this has the solution"

$$\boldsymbol{\alpha}' = [(r' - R_{in})^2 R_{out}^3 \;/ \{ r\, r'^2 (R_{out} - R_{in})^3 \}]\; \mathbf{diag}\, (\alpha_1 (r'_t - R_{in})^2, \alpha_2 r'^2, \alpha_3 r'^2 \}$$

On the surface where we are evaluating the results, r' equals r. Also both the permittivity and permeability are scalars in air. We, therefore arrive at (after some more algebra):

$$\boldsymbol{\varepsilon}' = \{ \boldsymbol{\varepsilon}_0 / \mathbf{R} \}\; \mathbf{diag} \{ (r' - R_{in}) / r', 1, 1 \}, \qquad \text{and}$$

$$\boldsymbol{\mu}' = \{ \boldsymbol{\mu}_0 / \mathbf{R} \}\; \mathbf{diag} \{ (r' - R_{in}) / r', 1, 1 \},$$

It may be noted that the permittivity { $\boldsymbol{\varepsilon}_0$ } and permeability { $\boldsymbol{\mu}_0$ } values in air shown above are actually the values in a vacuum. Although, this technically an error, it is very small[21]. It may also be noted that most references actually solve for "relative" permittivity { $\boldsymbol{\varepsilon}_r = \boldsymbol{\varepsilon} / \boldsymbol{\varepsilon}_0$ } and permeability { $\boldsymbol{\mu}_r = \boldsymbol{\mu} / \boldsymbol{\mu}_0$ }. Thus in those references, the air values are simply 1 and not seen in the equations at all.

4. **Spherical Invisibility Cloak Discussion**

The description normally given for these cloaks is that of a "hole" being created by the design that removes the interior volume from being seen from the outside by preventing the passive reflection of EM waves. While correct, that is only half of the story. The shell actually provides a complete separation of two electromagnetic domains into a cloaked region and an outside region. More precisely, a complete cloak would not only cloak passive objects from incoming waves, but also cloak active interior devices by preventing their EM waves from going out and being detected.

The interesting result found in the interior with active sources is the presence of **extraordinary electric and magnetic surface voltages**[22] at the inner shell boundary. In another source[23], it is explained that "the true physical explanation is that the normal fields at the inner boundary of the cloak act as delta functions, which are caused by infinite polarizations of the material at the inner boundary … ". That should raise the question of what in the he** does that mean. It was found that the problem is in the boundary conditions that were used at the inner surface. Specifically, In order for Maxwell's Equations with sources to have finite energy solutions within the cloaked volume the tangential components of both the electric and magnetic fields must vanish[24] at the inner boundary. This last reference goes on to say "…thus the single coating construction is insufficient for invisibility. In practice, even for cloaking passive objects, this may degrade the effective invisibility." Two solutions are offered by the author of the last reference for this problem. The first and simplest, is to add a perfectly conducting liner inside the inner surface, thus making the interior seem like a passive object. The second (if I understand it correctly) is to add another meta-metal shell to transform from the present inner shell surface to one with the correct boundary conditions. This they call, "double coating." I assume the 2 shells would not be in direct contact.

5. **Cylindrical Invisibility Cloak**[25]

The cylindrical cloak equations do not look much different from the Spherical ones. In cylindrical coordinates (ρ, φ, z }, the Boundary Conditions and transform are:

$$f(P_{out}) = P_{out} \qquad \text{and} \qquad f(P_{in}) = 0 ;$$

$$\rho' = \mathbf{R}\, \rho + P_{in} \quad \& \quad \varphi' = \varphi, \; z' = z \quad (P_{out} \geq \rho \geq 0) ;$$

and $\quad \rho = \{ \rho' - P_{in} \} / \mathbf{R} \quad \& \quad \varphi = \varphi', \; z = z' \quad$ {the inverse transform} ;

where: { $\mathbf{R} = (P_{out} - P_{in}) / P_{out}$ } .

The Jacobian in terms of the transformed coordinates is then:

$$\mathbf{J}(\mathbf{r}) = \rho^{-1}\{\rho' \mathbf{1'} + (P_{in}/r'^2)\,\mathbf{RR}^T\} + \mathbf{1''},$$

Where: $\mathbf{1'}$ = diag { 1, 1, 0 } & $\mathbf{1''}$ = diag { 0, 0, 1 } .

Similar to the spherical process the equations are evaluated at point { $\mathbf{P'}^T = [\rho', 0, 0]$ }. Which results in:

$$\boldsymbol{\varepsilon}' = \{\boldsymbol{\varepsilon}_0 / \mathbf{R}\}\ \mathbf{diag}\{\ \mathbf{R}^2,\ 1,\ 1\},\qquad \text{and}$$

$$\boldsymbol{\mu}' = \{\boldsymbol{\mu}_0 / \mathbf{R}\}\ \mathbf{diag}\{\ \mathbf{R}^2,\ 1,\ 1\},$$

6. Cylindrical-Spherical Cloak Interface Discussion

It was stated in section 2 that due to the derivative discontinuity the rays which pass through the discontinuity an outline of it will be seen but would probably be very faint. Now with the results of sections 3 and 5, there is a way to quantify that statement. (It may be noted that although this section is not particularly difficult to determine, the author did not find it in any reference.)

In any material, the index of refraction is actually a measure of the speed of light in that medium. Specifically, it is the ratio of that speed to the speed of light in a vacuum. Since the peed of light is also defined as the square root of the permittivity times the permeability of any medium, sections 3 and 5 provide us with a method of comparing the index in the spherical cap to that in the cylinder. As light moves from one medium to another, it is refracted. That is, its velocity and therefore its direction change. But in addition some of the light is usually reflected when it changes mediums. A measure of the fraction of the light reflected is provided by the “reflectivity” of the boundary. In the present case, the angle of incidence is normal (perpendicular to the boundary), thus making the reflectivity equation quite simple to determine:

$$R = (n_2 - n_1)^2 / (n_2 + n_1)^2.$$

Thus, a knowledge of the index of refraction of the two mediums allows a determination of the fraction (or percent) of the wave that is reflected.

$$n = \text{sqrt}\{\boldsymbol{\varepsilon\mu} / (\boldsymbol{\varepsilon}_0\boldsymbol{\mu}_0)\} = \boldsymbol{\alpha/\alpha_0},$$

where the last portion is due to the fact that in these equations the permittivity has the same value as the permeability. To compare the two results we first convert the cylindrical components to spherical coordinates.

$$\alpha_c(r) = \mathbf{R}_1(1 + \mathbf{R}_1^{-4})^{1/2},$$
$$\alpha_c(\theta) = 1 / \mathbf{R}_1,$$
$$\alpha_c(\phi) = \cos^{-1}[(1 + \mathbf{R}_1^{4})^{-1/2}].$$

Then, since b and a are the same in both problems, $\mathbf{R} = \mathbf{R}_S = \mathbf{R}_C$ thus yielding:

(radial)	$n_c / n_r = \mathbf{R}(1 + \mathbf{R}^{-4})^{1/2}$
(Rotational)	1 (no reflectance)
(axial)	$n_c / n_r = \cos^{-1}[(1 + \mathbf{R}^{4})^{-1/2}] / \mathbf{R}$

If the cloak thickness is 1% of the radial area being cloaked, this indicates that the reflectance will be 2.945% of the radial and 1.48% of the axial component. The total RMS reflectance is therefore 3.296% of the wave. Since this reflected energy cannot stay in the cloak, it will be ejected, thus providing the outline at the interface that was discussed in Section 2. This will occur to some extent at any interface. Thus, however the cloaking is done an outline will be discernable if one stares at it.

7. Semi-Triangular Shaped Invisibility Cloak

Since we have already determined, the spherical and Cylindrical Permittivities and Permeabilities, the triangle will be approximated those pieces. The top and bottom pieces will be portions of a large sphere cut into triangles. Along the sides, they will ne attached to three sections of (approximately) half cylinder cloaks cut to exactly the length of the triangle sides. The cuts will not be exactly at the half cylinder point since we would wish these pieces to fit exactly to the slight curvature of the top and bottom pieces. The corner pieces will be sections of a another spherical cloak. The cylindrical portions were cut to the lengths of the sides to allow a flat interface between each of them and the corner sections. The amount of spherical cloak needed for the corners obviously depends on the triangle angles. For simplicity, this report will only look at an equilateral triangle. Each interior angle is therefore 60°. Since each cylinder portions will be 90° each the end sections would be 120° of the sphere.

Before proceeding any farther, we have to assume some values for the triangle. Following a recent sighting, we will assume the triangle sides have a length of 50 feet and a height of 20 feet. We assume the radius of the top and bottom plates is 50 times larger than the length of the triangles side, Additionally we also want maintain a constant cloak thickness ($R_{out} - R_{in}$). Although we will start with a value of 1 foot throughout, the value with a minimum reflectance will be calculated. Although we know the equations for the permittivity and permeability of each section, each are in the values of written in the values of **R** for that portion. Since the radius of the corner sections and the side sections are the same, the value for their **R**s will remain equal. That however cannot be said for the top and bottom plates.

$\mathbf{R}_1$ (corner) = 0.9; $\mathbf{R}_1$ (sides) = 0.9, $\mathbf{R}_2$ (top/bottom) = 0.0008.

With the figures stated, one can also determine the effect of the spherical arc on the flatness of the top and button sections. In Cartesian coordinates, the sphere can be written:

$$a < x^2 + y^2 + z^2 < b .$$

There, therefore will be a maximum bulge of 1.5 inches in the center of each of the plates.

Using the variable $\boldsymbol{\alpha}$ to represent both the relative permittivity and permeability, we can write the equations for each section as:

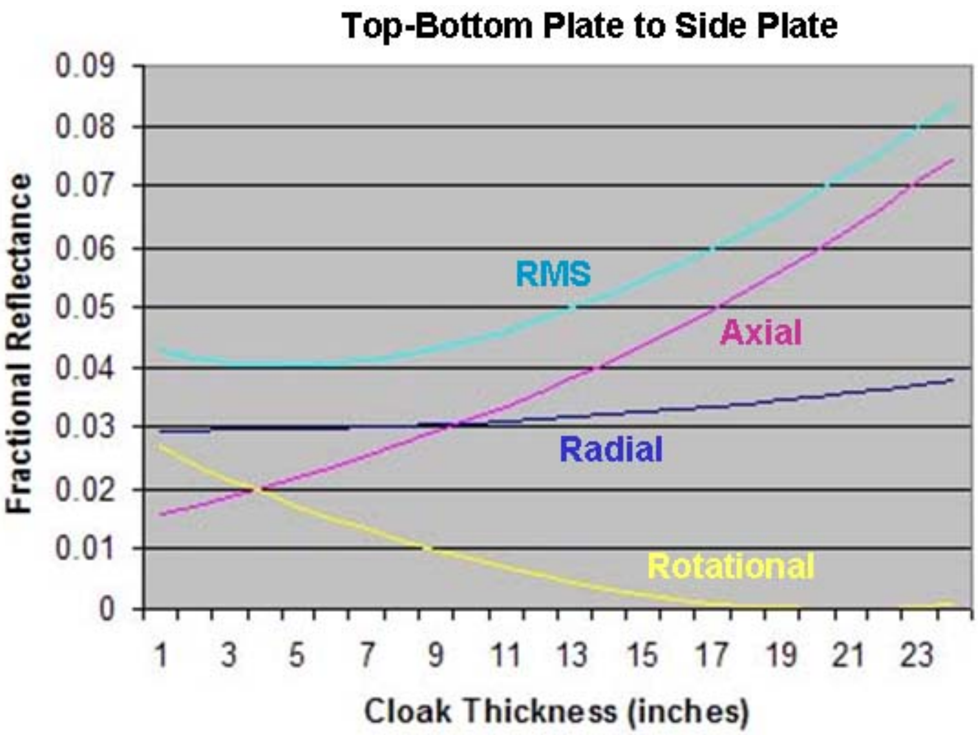
Corner Portions (r, θ, ϕ): $\boldsymbol{\alpha}_1 = \text{diag}\{ 1 ,\ 1 / \mathbf{R}_1, 1 / \mathbf{R}_1 \}$,
Sides (ρ, θ, z): $\boldsymbol{\alpha}_2 = \text{diag}\{ \mathbf{R}_1 , 1 / \mathbf{R}_1, 1 / \mathbf{R}_1 \}$,
Top/Bottom: (r, θ, ϕ) $\boldsymbol{\alpha}_3 = \text{diag}\{ 1 ,\ 1 / \mathbf{R}_2, 1 / \mathbf{R}_2 \}$.

In the same manner as seen in the previous section the side sections are converted to spherical coordinates and a reflectance can be calculated. The difference that occurs here is found in the top or bottom interface to the sides. Since the overall radius of the sections is not the same, the cloak fractional radius (**R**) is also not the same. The Index of refraction ratios for that interface are given by:

: (radial) $n_c / n_r = \mathbf{R}_1 (1 + \mathbf{R}_1^{-4})^{1/2}$

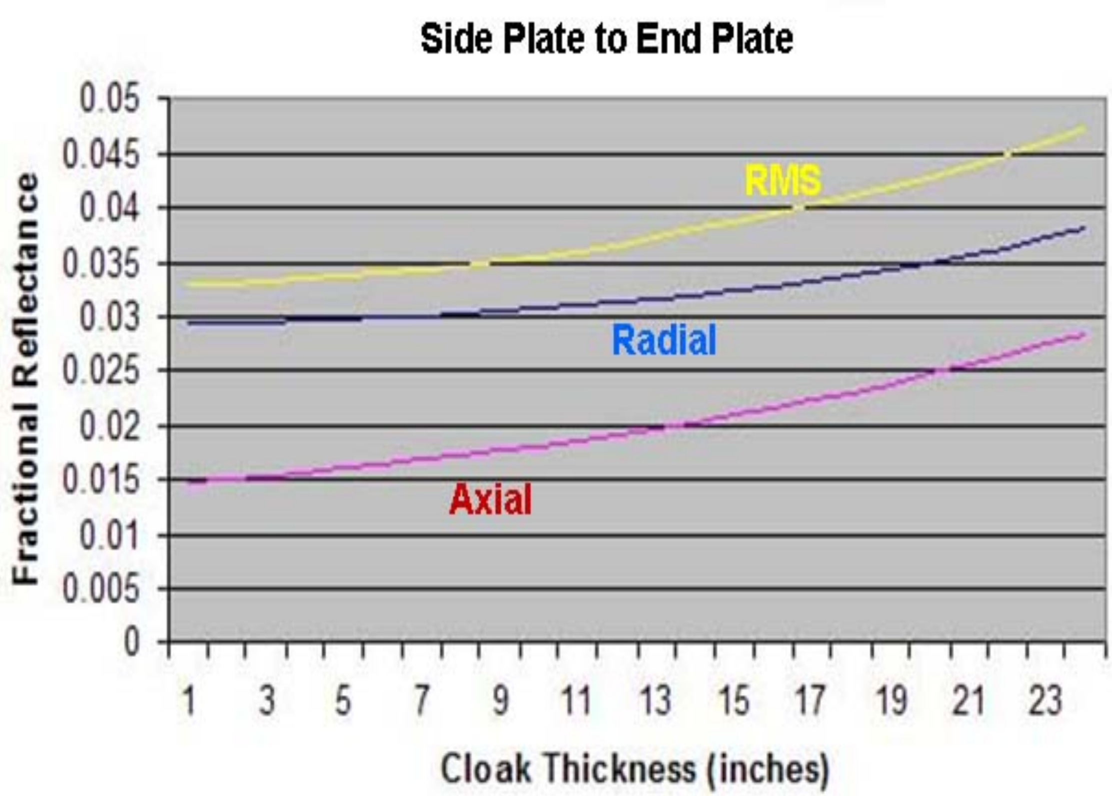
(axial) $n_c / n_r = \cos^{-1} [(1 + \mathbf{R}_1^{-4})^{-1/2}] / \mathbf{R}_2$

As can be seen, the radial portion has not changed but the axial one has. The following is a graph of the Radial, Axial, and RMS reflectances at the interface for the top-Bottom section and the side sections for various shell thicknesses.



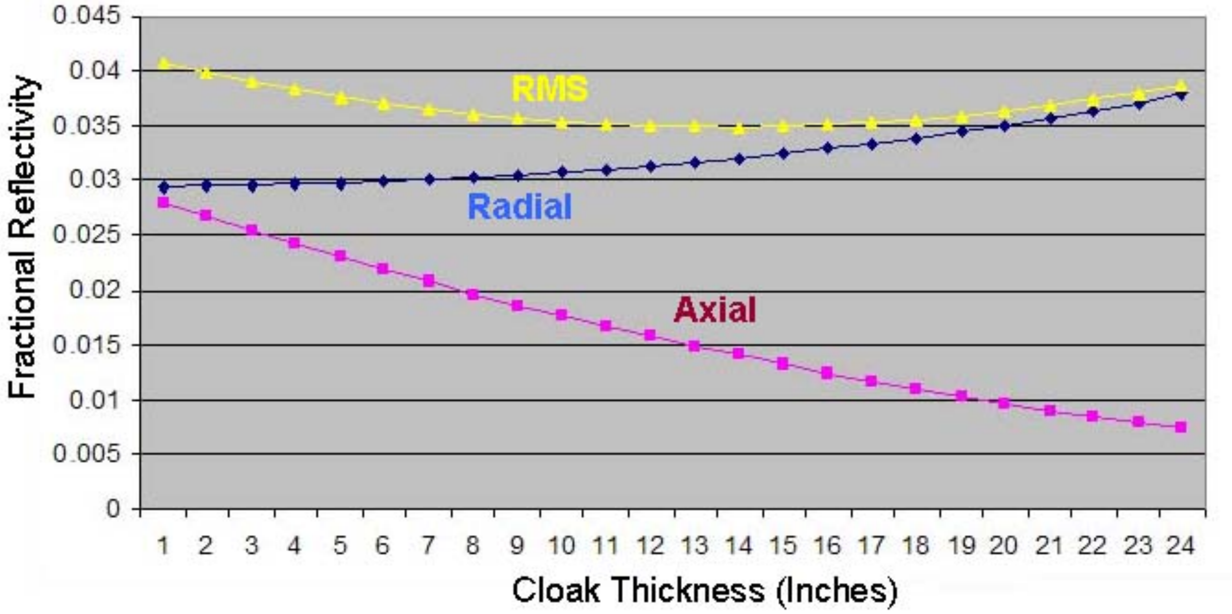
Therefore, for a cloak thickness of 1 inch, the following reflectances would occur.

	Radial	Rotational	Axial	RMS
Side to End Interface	2.9449%	0	1.4727%	3.2926%
Side to Top-Btm Interface	2.9739%	1.7129	2.1714%	4.0612%
End to Top-Btm Interface				

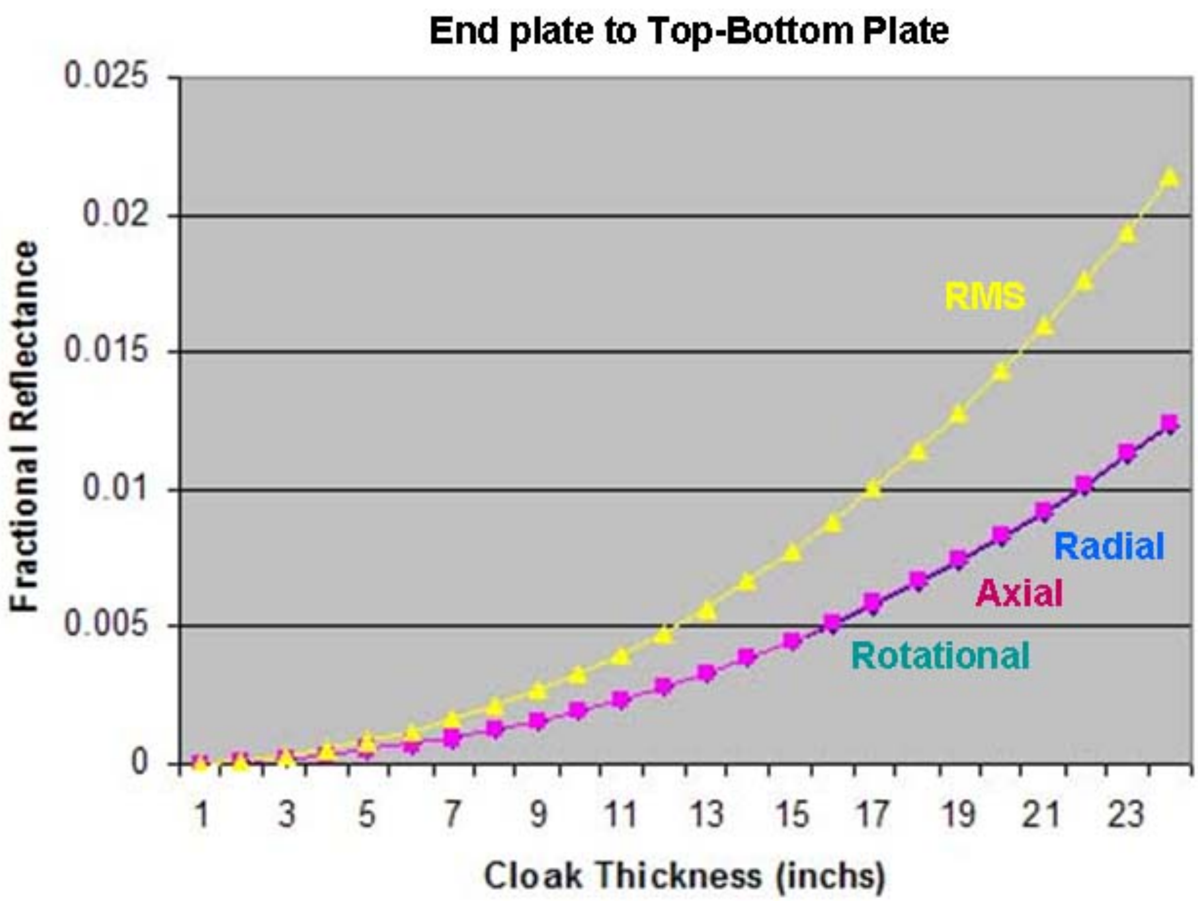


A similar graph has been calculated for the side to end interface and shows the same basic tendencies. Therefore if one were to create the triangular cloak with a thickness of 1 inch the resulting reflectance values for the interfaces would be:

	Radial	Axial	RMS
Side to End Interface	2.945%	1.479%	3.296%
Side to Top-Btm Interface	2.945%	0.0867%	2.943%
End to Top-Btm Interface	0.00175%	0.00175%	0.00175%



0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
Fractional Reflectivity
RMS
Radial
Axial
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
Cloak Thickness (Inches)



End plate to Top-Bottom Plate
0.025
0.02
0.015
0.01
0.005
0
Fractional Reflectance
RMS
Radial
Axial
Rotational
1 3 5 7 9 11 13 15 17 19 21 23
Cloak Thickness (inchs)

8. Invisibility Cloak Ray Tracing[25]

A complete report on the optics of Invisibility should conclude with a discussion of Ray Tracing. Ray Tracing is not a new field of study. Older books would call it Geometrical Optics.. Present day methodology uses Hamilton's equations of motion to determine the path of single frequency rays throughout any continuously varying inhomogeneous medium.

$$\partial H / \partial \mathbf{k} = \partial \mathbf{x} / \partial \tau \qquad \text{and} \qquad \partial \mathbf{k} / \partial \tau = -\partial H / \partial \mathbf{x}$$

In these equations, H is the Hamiltonian {Energy}, **k** is the relative wave vector in a medium { $\mathbf{k} = \omega \mathbf{v} / \omega c$ }, **x** is the path distance, and τ is a variable (essentially time) which parameterizes the paths. Since we are only using this for a single frequency, we can write the fields

$$\mathbf{E} = \mathbf{E}_o \exp\{ i (\mathbf{k}x - \omega t) \} \qquad \text{and} \qquad \mathbf{H} = \mathbf{H}_o \exp\{ i (kx - \omega t)$$

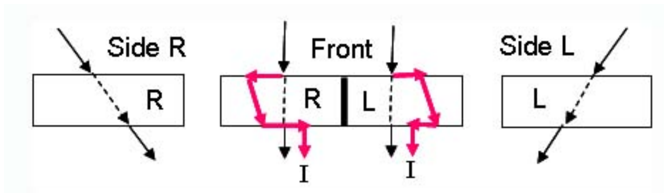
When discontinuities are encountered (cloak boundaries) wave matching is used. We have to determine $\mathbf{k}_1$ (wave vector in medium 2) from our knowledge of $\mathbf{k}_1$ and prosperities of the mediums. The transverse components are found by requiring conservation of the transverse eave vector is conserved across the boundary. The normal component is determined by solving for $\mathbf{k}_1(n)$ that will satisfy the plane wave dispersion relation ω(k). (Essentially, these requirements are a version of Snell's law.) In all cases, the path of the ray can be determined from ($d\mathbf{x} / d\tau$). It may be noted that Hamiltonians are usually quadratic and therefore provide two solutions. The correct one will always be the one which gives a positive result for the path.

The process of tracing any ray starts with the assignment of location and initial conditions. The ray is then traced to its intersection with some portion of the cloak. In air, this will be a straight line. The ray is refracted into the cloak. This provides the initial conditions to integrate Hamilton's Equations to follow t through the cloak. If it intersects another boundary, the same steps are followed again. This will continue until the ray intersects the outer boundary for the second time and is refracted out. That refraction yields the initial conditions and location for the ray's new path in air. The difficulty is in the integration of Hamilton's equations. That is normally done on a computer. Since this author does not have access to the normal integration routines, it will not be done here.

The imaging of an object beyond the triangle requires some thought. It is well known where objects image if the sphere that was cut to make the top and bottom plates were continuous. Aside from the ray that goes through the sphere center-point, all rays will go directly through the sphere and image location would not be changed. For the triangle, the cut cylinder will transfer the ray exactly the same as if the sphere had not been cut. The ray in the triangle analogues to the rays through the center of the sphere is the ray that traverses the top to bottom midline of the triangle.

What was written is would be exactly true except for a couple of items. One off those items one will have a small effect and the other a larger one. The small effect is due to motion of the triangle or sphere. Movement causes a variation of location between the entry point and the exit point. However, since, the triangle's (or sphere's) speed would be small compared to that of light; the offset probably would not be noticed.

The larger effect is a result of the angles provided by the triangle. Any ray not normal to the top plate of the triangle will travel a distance "x" through the top plate before coming to the cylindrical portion. It will then travel around to the bottom plate. The problem occurs due to the fact the entrance into the bottom plate is not in the same location as the exit from the top plate. That is shown in the following figure. The lines through the sides are shown tilted since they will be following the same angle as the incident ray. Thus the ray in the "R" section comes out a little closer to the centerline; and the one in the "L" section a little farther from the centerline.



Two incident rays are shown in this figure. The ray on the side titled “R” is tilted toward the front of the triangle and the one on the side titled “L” is tilted toward the back. In both cases, the black ray through the side and out the bottom is where the ray should be to see the star or object in the correct location. Let the rays that go from the initial entry points to the sidewalls be a distance “d”. Since the rays are tilted, they pass through the sidewalls on a tilt. When they exit from the sidewalls, they also travel a distance “d” to the exit point. However since they started the lat leg at a different location on the sidewall, they do not come back to the initial ray location.

The above illustration shows both exit rays moved to the right. They can actually move in any direction depending on the tilt of the incident ray.

9. Conclusions

The study of invisibility cloaks only started in the middle of the last decade. In the few years since that start, we are almost at the point of having that capability of cloaking a flying object. It will take some engineering breakthroughs and a great deal of money but it is obviously doable. Since I believe we are that close, how much further advanced would another race have to be to already have a working cloak.

The most interesting results found in the above is how well they tally with recent sightings reported to MUFON. To cloak any reasonable shaped engineered device will require a piece-wise continuous cloak. It has been shown that this always has some reflectance at the interfaces. Therefore, any reasonable cloak will have at least some vague outline of its shape. Although it was not specifically stated in this report, any reasonable machine has to have a way for the people inside to see out and navigate. It also has to have a way to dissipate heat. Both of those requirements demand a non perfect cloak. It was also shown that with piecewise continuous cloaks, the image of items behind the cloak will be moved from the actual locations.

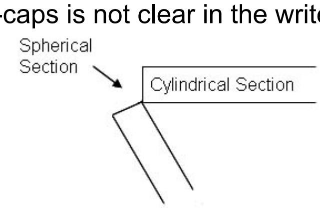
1. J. Pendry, D. Schurig, and D. Smith, "Controlling electromagnetic fields"; Science 312, 1780-1782 (2006).
2. Conformal Transform: A mapping or transform preserves angles.
3. Meta-materials are nano-engineered man-made materials that obtain their prosperities from slight inhomogeneities in structure rather than composition.
4. Anisotropic Material: The material properties depend on the direction.
5. The Electric Permittivity of a medium is a measure of the response of a medium to an impressed Electric Field. In general it is not a constant. Specifically for this report it can depend on: material the medium is composed of and the frequency of the impressed electric field.
6. The magnetic Permeability is a measure of the response of a medium to an impressed Electric Field. In general it is not a constant. Specifically for this report it can depend on: material the medium is composed of and the frequency of the impressed electric field.
7. Isotropic Material: The material properties identical in all directions.
8. Dielectric Material: A dielectric is a polarizable electrical insulator.
9. The Refractive Index is a measure of the speed of light in a material compared to its value in a vacuum. It is related to the Permittivity and permeability via the equation { $n = \sqrt{\varepsilon_0\mu_0 / \varepsilon\mu}$ }.
10. J. Li, J. Pendry; “Hiding under the carpet: a new strategy for cloaking”; Physics Review Letters; **101**; 203901; (2008)
11. J Tyson; Quasiconformality and Quasisymmetry in Metric Measure Space; Annals Academiæ Scientiarum Fennicæ, Mathematics; **23**; 525–548; (1998)

12. X Chen, Y Luo, J Zhang, J Klye, j Pendry, S Zhang; “Macroscopic Invisibility Cloaking of Visible Light”; Nature Communications; **2** (2); 176; (2011)
13. B Zhang et. al.; “Macroscopic Invisibility Cloak for Visible Light”; PRL; **106** (3); 033901; (2011); (open access copy ; www.nature.com/ncomms/journal/v2/n2/full/ncomms1176.html)
14. An invisibility Carpet Cloak is basically a 2-d cloak under which a stationary object can be hidden.
15. A Transverse Eclectic (TE) mode is a mode whose Electric Field vector is normal to its direction of travel (propagation direction). A Traverse Magnetic mode is the same with Magnetic Field vector substituted for Electric Field vector. These specific modes normally occur due to boundary conditions of waveguides. A general wave will be a combination of the modes.
16. The Jacobian matrix (in terms of the original coordinates) with this error would be:

$$\hat{\mathbf{J}} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \frac{\cos\theta\cos\phi}{r} & \frac{\cos\theta\sin\phi}{r} & -\frac{\sin\theta}{r} \\ -\frac{\sin\phi}{r\sin\theta} & \frac{\cos\phi}{r\sin\theta} & 0 \end{pmatrix}.$$

This error was made in a doctoral thesis document found on the web. “Theory of transformation optics and invisibility cloak design”; KTH; Stockholm, Sweden; 2011

17. In matrix notation a 3-d Identity tensor has 1’s down the diagonal and 0’s off diagonal. It is obtained from the partial derivative of **R** with r.
18. A tensor product of 2 vectors is normally written ($\mathbf{a} \otimes \mathbf{b} = \mathbf{ab}^T$). In 3-d Matrix notation the elements of this matrix are: row 1 = {a_1b_1 a_1b_2 a_1b_3); rows 2 and 3 are the same with a_1 changing to a_2 and a_3 respectively.
19. A diagonal matrix is one with elements on the diagonal and zeros elsewhere. In terms of the Jacobian, it means r’ is only a function of r, θ’ is only a function of θ, and ϕ’ is only a function of ϕ.
20. The product of diagonal matricides is a diagonal matrix with products of like elements on the diagonal. Ie **diag** (a_1, a_2, a_3) **diag** (b_1, b_2, b_3) = **diag** (a_1b_1, $a_2 b_2$, $a_3 b_3$)
21. The air value for the electric permittivity is ~1.0006 times that of a vacuum. The air value of the magnetic permeability is ~1.0000004 times that of a vacuum.
22. Zhang, B., Chen, H.,Wu, B.-I. and Kong, J. A. (2008a). “Extraordinary Surface Voltage Effect in the Invisibility Cloak with an Active Device Inside. Physical Review Letters, 100 (6), 063904 (PRL normally only contains an outline of a methodology and the results. It is rarely more than a page or so long. Complete copies of this original document are available for free on the net.)
23. M Yan, W Yan, M Qiu; “Invisibility Cloaking by Coordinate Transformation” ; Progress in Optics; **52**; Chapter 4; 261-301; (2009)
24. A. Greenleaf, Y Kurylev, M Lassas, G Uhlmann; “Full-Wave Invisibility of Active Devices at All Frequencies”; Comm. Math. Physics; **275**, no. 3; 749-789; (2007)
25. M Crosskey, A Nixon, L Schick, G Kovacic; “Invisibility Cloaking via Non-Smooth Transformation Optics and Ray Tracing”; Physics Letters A; **375**; 1903-1911; (2011)
Note: Many authors do the spherical and cylindrical cloaks. This source is just considered more complete.
26. If description of the spherical end-caps is not clear in the write-up:



Mutual UFO Network

Invisibility Cloaking
via
Coordinate Transform Optics

(A review of the literature covering Invisibility Cloaking with specific reference to Triangular Craft)

by CC Paulson